Application of a New Effective Viscosity Model to Turbulent Plane Couette Flow

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Nomenclature

A,C = constants occurring in semi-logarithmic laws for turbulent layers
c_f = skin-friction coefficient, $\tau_w/\frac{1}{2}\rho u_c^2$
h = half width of channel
Re = Reynolds number based on half width, hu_c/ν
S = dimensionless centerline velocity gradient, $(h/u_c)du/dy)_c$
u = velocity in streamwise direction
u^* = friction velocity, $(\tau_w/\rho)^{1/2}$
u^+ = dimensionless velocity, u/u^*
U = relative velocity between planes of the Couette flow system
y = coordinate normal to wall
y^+ = dimensionless wall distance, yu^*/ν
z = dimensionless wall distance, y/h
γ = dimensionless ratio of effective to kinematic viscosity
ρ = fluid density
τ = shear stress
ν = kinematic viscosity

Subscripts

b = edge of viscous sublayer
c = centerline of Couette flow system
o = edge of overlap layer
w = wall value

Introduction

FLOW between parallel planes in relative motion is commonly referred to as Couette flow although Couette himself studied the flow between concentric cylinders. Numerous attempts to provide a theoretical model for turbulent plane Couette flow[1–7] have failed to predict both accurate velocity profiles and skin friction. Work done prior to 1959 has been summarized by Robertson.[7] At the present, Refs. 6 and 7 provide the bulk of the experimental data.

The purpose of this Note is to apply a simple effective viscosity model formulated in Ref. 8 to turbulent plane Couette flow.

Analytical Description of Turbulent Plane Couette Flow

The effective viscosity model is represented by the equations

$$\nu_{\text{eff}} = \nu, \quad 0 \le y^+ \le y_b^+$$

$$\nu_{\text{eff}} = \nu\left[1 + \frac{y^+ - y_b^+}{y_0^+ - y_b^+}(\gamma - 1)\right], \quad y_b^+ \le y^+ \le y_0^+ \quad (1)$$

$$\nu_{\text{eff}} = \nu\gamma, \quad y^+ \ge y_0^+$$

These relations hold in the viscous sublayer, the overlap layer and the outer layer, respectively.

For Couette flow, the Navier-Stokes equations show that the shear stress is constant. The shear stress and effective viscosity are coupled by the Newton-Boussinesq equation

$$\tau/\rho = \nu_{\text{eff}}\, du/dy$$

Integrating, and making use of the appropriate boundary conditions for each region, we obtain the velocity profiles

Sublayer

$$u^+ = y^+ \quad (2)$$

Received February 20, 1970; revision received June 18, 1970.
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Table 1 Comparison of present predictions with Reichardt's measurements

	Present predictions		Reichardt's measurements air		water	
Re	α	u_c^+	α	u_c^+	α	u_c^+
17,100	716	23.9	723	23.7		
14,650	624	23.5	629	23.3		
10,850	479	22.7	476	22.8	480	22.6
9,600	430	22.3	438	21.9		
9,500	426	22.3			423	22.4
7,750	357	21.8	358	21.6		
7,500	346	21.7			346	21.7
5,800	276	21.0			277	21.0
5,786			278	20.8		
4,930	240	20.6	236	20.9		
2,955	153	19.3	146	20.2		

Overlap layer

$$u^+ = A\ln\left[1 + \frac{y^+ - y_b^+}{A}\right] + y_b^+ \quad (3a)$$

or

$$u^+ - u_c^+ = A\ln(z + \beta) - C \quad (3b)$$

Outer layer

$$u^+ - u_c^+ = -\frac{1}{\gamma}[y_c^+ - y^+] = -\frac{y_c^+}{\gamma}(1 - z) \quad (4)$$

where y_b^+, A, C, and β are parameters to be determined, and

$$A = y_0^+ - y_b^+/(\gamma - 1)$$

$$u_c^+ = u_c/u^*$$

$$y_c^+ = hu^*/\nu$$

These equations must satisfy the following compatibility and matching conditions[8]: 1) In the overlap layer both the inner and outer velocity profiles should give the same information. 2) At the points y_b^+ and y_0^+ the velocity profiles should be continuous in both magnitude and gradient. 3) The coordinates y^+ and z should be directly proportional to each other.

To plot the velocity profiles given by Eqs. (2, 3, and 4), values are needed for A, y_b^+, u_c^+, y_c^+ (or α), γ and C. The matching condition introduces a further unknown in z_0 or y_0^+. However, it also provides two simultaneous equations which can be solved for two of these seven unknowns. By introducing the flow Reynolds number, which is independently specified, the unknowns may be reduced by a further two. Thus, knowing the Reynolds number and three of the unknowns (A, C and y_b^+) we can solve the matching equations and obtain a complete description of the velocity profiles. The complete procedure is given in Ref. 8.

Determination of the Universal Constants A, C and y_b^+

To fit the model to turbulent plane Couette flow, two sets of data measured in water and reported in Ref. 9 were used. The apparatus used consisted of a twin belt system separated by a test channel 1.25 in. wide. The belts were driven by two

Table 2 Results predicted by present analysis

Re	6000	9150	Re	6000	9150
c_f	0.0045	0.00405	α	284	412
u_c^+	21.1	22.2	γ	43.3	63.2
z_o	0.458	0.463	β	−0.0121	−0.00835
y_o^+	131.4	190.5	$S = \left.\frac{h}{u_c}\frac{du}{dy}\right)_c$	0.314	0.295

pairs of 3-in. diameter spindles on 25-in. centers. Thus the central plane velocity was maintained at approximately one-half the relative velocity between the two belts. With only a single moving belt,[6,7] a midplane velocity about 0.3 of the velocity of the running belt has been observed.

As profiles are measured in the $(u/u_c) - z$ coordinates, values of skin friction are required to convert them into $u^+ - y^+$ coordinates. These were determined from the Clauser plot[10] using the approximate equations in the inner region given by Reichardt and Robertson.[7] The values of c_f were found to be 0.0044 and 0.0041 for Reynolds numbers of 6000 and 9150, respectively. The values were used to nondimensionalize the corresponding profiles. The transformed data are depicted in Fig. 1.

We now seek to fit an equation to the measured profile in the semilogarithmic or overlap layer. The value $A = 2.96$ is obtained from the slope of the linear portion of the profile. A good fit of the data in this region was achieved with $y_b{}^+ = 6.4$. The divergence parameter, C, determined by the maximum deviation at the edge of the outer layer, is 1.2.

On substituting these parameters into the profile equations derived in the preceding section, the skin-friction relation can be obtained by straightforward algebraic manipulation.

$$(2/c_f)^{1/2} = 2.96 ln\ [Re(c_f/2)^{1/2}] + 4.39 \qquad (5)$$

For the given Reynolds numbers, the values of c_f were found from Eq. (5) to be 0.0045 and 0.00405 at Reynolds numbers of 6000 and 9150, respectively. These values of c_f have been plotted in Fig. 2, together with those taken from Ref. 7, for comparison with previously reported data for Couette flow systems.

Discussion

The skin-friction relation given by Eq. (5) is intermediate between the results of Robertson and Reichardt. The data attributed to Reichardt were reduced by Robertson from Ref. 5 using the velocity gradient at the wall computed from the velocity profiles. This points to the relative inaccuracy of these profiles close to the wall. Reichardt has also obtained c_f measurements experimentally. These are reported in a later publication (see Ref. 6). A partial comparison of these results with the present prediction is given in Table 1. These measurements agree very well with the present prediction.

Since Reichardt has insured a midplane velocity of 0.5 U, two of Reichardt's profiles (at Reynolds numbers of 1450 and 17,000) have also been plotted in Fig. 1. The c_f values used for nondimensionalizing the velocity and the wall distance were calculated from Eq. (5). The degree of agreement is fair in the outer portion. However, in the inner portion, the agreement becomes rather poor. These two profiles even appear to be inconsistent since in one case the velocity profile is higher than the present correlation of the law of the wall while the opposite is true in the other case. This feature tends to confirm the statement on the poor condition of Reichardt's profiles close to the wall made earlier.

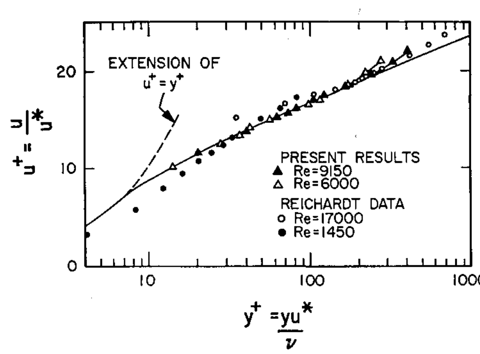

Fig. 1 Experimental velocity profiles plotted in law of the wall coordinates, with empirically fitted law of the wall.

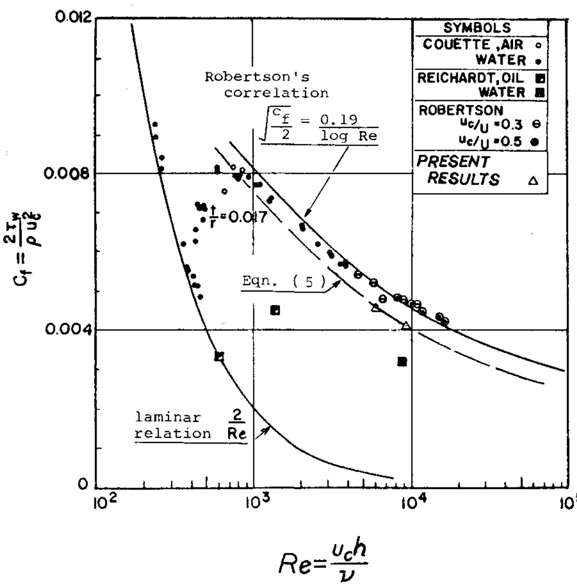

Fig. 2 Skin-friction coefficient as function of Reynolds number in Couette flow.

Robertson's data were mainly obtained at a midplane velocity equal to 0.3 U. They have not been included because they seem to be characteristic of data obtained in a flow that is not fully developed. This behavior is evidenced by the facts that the nondimensional velocity profile is fuller and the skin friction obtained from the Clauser plot is higher than for fully developed flows.

In Fig. 2, the data of Robertson obtained in a plane system appear to be a continuation of the data of Couette obtained in a cylindrical system, even though these two cases are not physically identical. The presence of centrifugal forces in the latter would cause skin friction to be higher, as evidenced from the laminar data. (A summary of other parameters predicted by the present method is given in Table 2.)

Conclusions

The effective viscosity model of Chue has been applied to the case of turbulent plane Couette flow. The simple model accurately predicts both the velocity profile and shear stress values. Using the authors' data and data of Reichardt, a skin-friction relation has been obtained. This shows the same trend as Robertson's correlation, but gives slightly lower friction. However, values obtained from this relation are in good agreement with data of Reichardt.

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Rotational Temperature Measurements in Low-Density Flows

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Introduction

THE electron beam fluorescence technique has been described by Muntz.[1] A narrow beam of electrons is passed through the gas and from the intensity of fluorescence the density can be determined, whereas for nitrogen the rotational temperature can also be obtained from the relative intensities of rotational lines in the spectrum. Muntz[1] proposed a mathematical model which relates the measured line intensities of the first negative system of nitrogen to the population of the rotational levels in the unexcited gas. If this population distribution is Maxwellian, then the rotational temperature of the gas can be calculated.

Experimental studies have shown that although density measurements are reliable,[2] the rotational temperature calculated using Muntz's model is higher than the true temperature.[3] In an earlier Note,[4] this discrepancy was tentatively explained in terms of excitation by secondary electrons, although there is still some doubt about the exact process involved. Using Ashkenas'[3] experimental results, Muntz's theory was modified to include the assumed secondary electron excitation leading to the expression for the relative line intensities

$$I(K')/(2K'[G_p + n \cdot G_s]) \; \alpha \; [e^{-\phi K'(K'+1)/T_R} \qquad (1)$$

where $I(K')$ is the relative line intensity and K' is the rotational quantum number. G_p is a function of K' and T_R given by Muntz[1] for the primary electrons and G_s is a similar but unknown function for the secondary electrons. The number density is denoted by n, whereas $\phi = 2.88$ for the ground state vibrational level.

These electron beam fluorescence techniques have been used to study rarefied flow over a sharp leading edge flat plate. From the measured density and the rotational temperature calculated by using Eq. (1), the static pressure in the shock layer was obtained. The variation through the shock layer for the weak interaction and merged regimes was examined and the pressure in the viscous layer compared with the measured surface pressure corrected for orifice effects.

Received February 4, 1970; revision received July 27, 1970.

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Experimental Details

The experiment was performed in the NPL (National Physics Laboratory) Low-Density Tunnel using pure nitrogen at a stagnation temperature of 632°K. The freestream Mach number, calculated from the ratio of the stagnation pressure to freestream pitot pressure corrected for viscous effects, was 6.28 and the Reynolds number was 370 per cm. A water-cooled model was mounted in front of the electron gun and joined to it by a narrow drift tube. The electrons pass along this tube and emerge through a 0.5-mm-diam hole in the model, normal to the surface. More details are given in Refs. 2 and 5. In all tests the pressure in the drift tube was maintained equal to that on the surface of the model to eliminate any flow of gas through the tube.

An optical system produced an image of the beam at the entrance slit of the spectrometer orientated so that the length of the beam was perpendicular to the slit. The spectrometer could resolve all the lines in the 0–0 band of the first negative system of nitrogen sufficiently for the peak intensity to be proportional to the line intensity. A portion of the light beam incident on the entrance slit was deflected by a small mirror onto a photomultiplier so that the output was proportional to the total intensity of fluorescence and served as a monitor for variations in flow conditions or beam current while measuring the line intensities of the 0–0 band.

Density and rotational temperature profiles through the shock layer were obtained at several positions along the centre line of the model. Density profiles were obtained in the usual manner[2] by measuring the beam current and the intensity of fluorescence at several points through the shock layer. At each point the relative intensity of the first 11 lines of the 0–0 band were also measured by rotating the grating.

Analysis of Rotational Spectra

If Eq. (1) is correct a plot of $\log_e$ of the left hand side against $K'(K'+1)$ will be a straight line, the slope of which is ϕ/T_R. A problem arises when applying this equation since G_s is a function of T_R and in Ref. 4 values of G_s/G_p could only be obtained at 78°K and 289°K. However, for the first 11 lines G_s/G_p does not vary strongly with T_R over this range, and therefore, can be assumed to vary linearly without making serious errors.

Except when near the surface of the model, the measured intensities gave a good linear plot although in some cases there was a small systematic increase in temperature as the number of lines used to calculate the temperature was reduced, possibly due to inaccuracies in G_s. The gradients in the flow regime investigated do not appear to be large enough to cause any nonequilibrium between rotational and translational degrees of freedom.

Within 2 or 3 static mean free paths of the surface the plots become very nonlinear. Assuming that the rotational and translational degrees of freedom are in equilibrium, this indicates that the velocity distribution function is non-Maxwellian, as might be expected if there is an appreciable temperature jump at the surface. Measurements were made to within half a mean free path of the surface and attempts were made to analyze the rotational spectra in terms of a two-stream Maxwellian model. In all cases the rotational temperature computed for the incident stream increased with the number of lines used in the calculation to such an extent that the values could not be regarded as reliable.

Rotational temperature profiles obtained using all eleven lines are shown in Fig. 1. If on increasing the number of lines used to calculate T_R from 7–11 there was a systematic increase in the temperature of more than 10%, the data was